

Technion - Israel Institute of Technology
Faculty of Mechanical Engineering

Theory of Vibration (034011)

Numerical Analysis of Dynamical Systems

- (i) Time Series
- (ii) State Space
- (iii) Poincare' Maps

Using:

Matlab Simulation Procedure

- (i) Simulation
- (ii) Sampling
- (iii) Graphics

Compiled By:
Sharon Pragai
Dr. Oded Gottlieb

October 2003

Matlab Procedure: Dynamical System Simulation

Simulation of a dynamical system is obtained by numerical integration of a system of 1st order differential equations with a corresponding set of initial conditions. Selection of the integration routine (e.g. Runge-Kutta or Gear solvers) is based on the characteristics of the dynamical system (e.g. smooth or stiff) and if automatic control of numerical error is required (e.g. adaptive integration scheme).

The outcome of the simulation consists of the response time series. We emphasize that dynamical systems subject to periodic excitation should be analyzed at steady state that is to be determined. Consequently, the outcomes of a dynamical system simulation are:

- i) time series ($x(t)$);
- ii) state space projections ($y(x)$) at steady state.

The Matlab function ODE45 is an adaptive solver that integrates a system of ordinary differential equations using a 4th order Runge-Kutta formula (with a 5th order correction).

In order to obtain on-line help information, just type "help ode45" when inside the Matlab window.

The Matlab command is:

$$[T, Y] = \text{ODE45} (@F, \text{TSPAN}, Y0, \text{OPTIONS})$$

where:

'F' - is the name of the file with the state space differential equations.

TSPAN - Tspan = [T0, Tfinal], initial and final integrating times.

If we look for a solution at specific times (T0, T1, T2, ..., Tfinal), define

Tspan = [T0 T1 T2, ... Tfinal], and $T_i = t_0 : \Delta t : t_f$.

Y0 - is a **column** vector of initial conditions.

OPTIONS - contains the optional values for the desired accuracy like:

relative error tolerance 'RelTol' ($1e-3$ by default) and

vector of absolute error tolerances 'AbsTol' (all components

$1e-6$ by default). Omitting OPTIONS argument will choose default values .

Example

Solve the second order differential equation:

$$\frac{d^2x}{dt^2} + \delta \frac{dx}{dt} + \alpha x + \beta x^3 = \gamma \cos(\omega t) \quad (1)$$

with the following initial conditions: $x(0) = x_0$; $\frac{dx}{dt}(0) = dx_0$

In time interval $t = 0 \div 150$.

This is equivalent to a system of two first order equations, 5 parameters, and two initial conditions:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= \gamma \cos(\Omega t) - \beta x_1^3 - \alpha x_1 - \delta x_2 \\ x_1(0) &= x_{10} \\ x_2(0) &= x_{20} \end{aligned} \quad (2)$$

In order to solve the problem:

1. Create two separate external m-files:
 - (a) "system1.m": stores the information about the equation set structure.
 - (b) "solve.m": includes the remaining data and necessary commands.
2. Type "solve" from the matlab window and the file "system1" will be used automatically. The results are stored in vector t and matrix x.
3. Plot solution graphs. The corresponding matlab command is "plot(x,y)" where x is an argument column-vector (horizontal-axis), while y is a function column-vector (vertical axis). File "solve.m" includes all the necessary commands, which shows how to plot the graphs.

```
% File "system1.m" - Dynamical System
```

```
function xprime = system1 (t,x)
```

```
xprime=zeros(2,1);
```

```
% ===== Parameter Definitions =====
```

```
alpha=-1.0;
```

```
beta=1.0;
```

```
gamma=0.2;
```

```
delta=0.3;
```

```
w=1.2;
```

```
% ===== State Space =====
```

```
xprime(1)=x(2);
```

```
xprime(2)=gamma*cos(w*t)-alpha*x(1)-delta*x(2)-beta*x(1)^3;
```

```
% File "solve.m" - Matlab Runge Kutta 4-5
```

```
close all; clear all; clc;
```

```
t0=0; % initial time
```

```
tfinal=150; % final time
```

```
x0=[0.5 0]'; % column vector of initial conditions
```

```
tspan=[t0 tfinal]'; % tspan can contain other specific points of integration.
```

```
option=odeset('RelTol', 1e-8, 'AbsTol', 1e-8);
```

```
[t,x]=ode45(@system1, tspan, x0, option);
```

```
% plotting the results
```

```
% =====
```

```
% Free Vibration: gamma = 0:
```

```
figure(1)
```

```
subplot (2, 1, 1)
```

```
plot (t,x(:,1)); grid;
```

```
xlabel ('t [sec]'); ylabel ('x(1) [m]');
```

```
subplot (2, 1, 2)
```

```
plot (t,x(:,2)); grid;
```

```
xlabel ('t [sec]'); ylabel ('x(2) [m]');
```

```
% Forced Vibration: gamma = 0.2:
figure(2)
subplot (2, 1, 1)
plot (t,x(:,1)); grid;
xlabel ('t [sec]'); ylabel ('x(1) [m]');
% the steady state space is defined by the last 20 percent of the solution vector.
% n is the total number of time nodes during integration
n=size(t);
subplot (2, 1, 2)
plot (x(round(0.8*n):n,1), x(round(0.8*n):n,2)); grid;
xlabel('x(1)[m]'); ylabel('x(2) [m/s]');
```

Numerical solution of $x'' + \delta x' + \alpha x + \beta x^3 = \gamma \cos(\omega t)$

(1) Free vibration: $\gamma = 0$

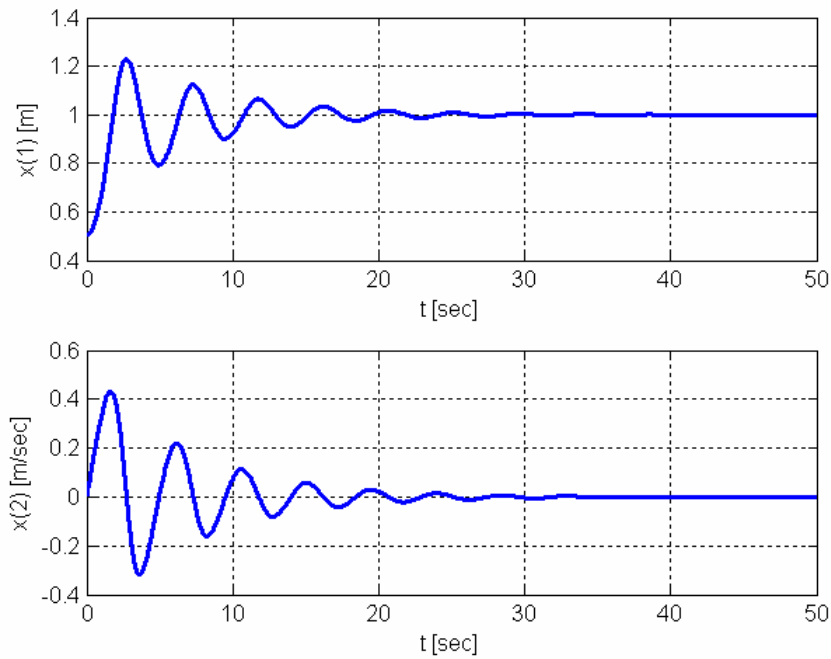


Figure 1: Free vibration response: $\gamma = 0$, $\alpha = -1$, $\beta = 1$, $\delta = 0.3$, $\omega = 1.2$. a) $x_1(t)$. b) $x_2(t)$.

(2) Forced vibration response: $\gamma = 0.2$

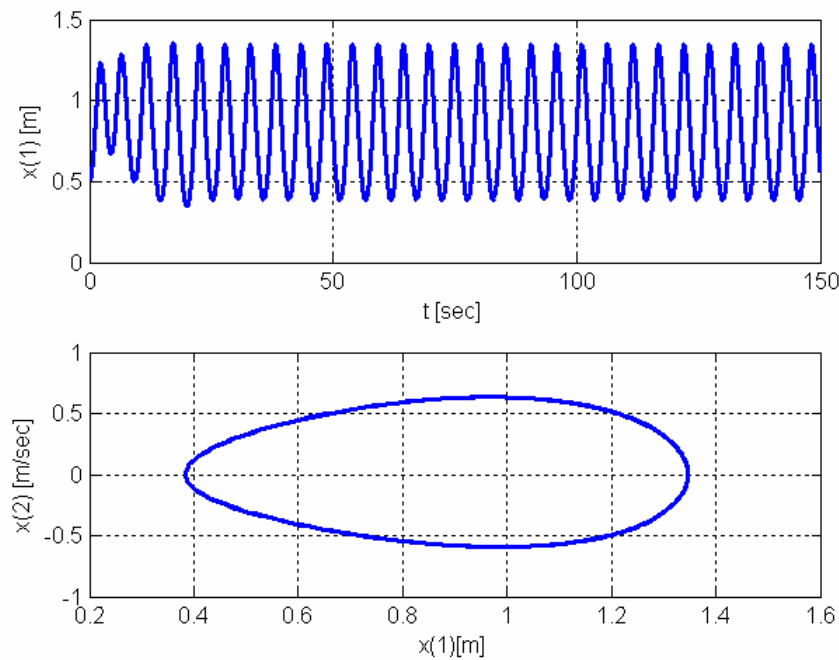


Figure 2: Forced vibration response: $\gamma = 0.2$, $\alpha = -1$, $\beta = 1$, $\delta = 0.3$, $\omega = 1.2$.

- Time series
- Phase plane at steady state ($120\text{sec} < t < 150\text{sec}$)

Poincare' Map for Periodically Excited Forced Vibration

A Poincare' Map consists of a sequence of points in state-space generated by the penetration of a continuous evolution trajectory through a generalized plane in space. For periodically excited dynamical systems, a Poincare' Map can be obtained by stroboscopically sampling the system state every forcing period (H. Poincare' 1854-1912).

Poincare' Maps are used to characterize stability of periodic system response (e.g. whether the response is periodic with the same period as that of the excitation).

The Poincare' Map of a periodically excited dynamical system is constructed by sampling system response at a fixed interval that equals the (minimal) period of the excitation.

A system with periodic response has a finite number of Poincare' points in state space:

- i) a single point: periodic response with the same period of excitation (T).
- ii) m points: subharmonic response (of order m) corresponding to periodicity with an integer multiple of the exciting period (e.g. mT , $m > 1$).
- iii) m points: ultrasubharmonic response (of order m/n) corresponding to periodicity with an integer submultiple of the exciting period (e.g. mT/n , $m, n > 1$).

A system which exhibits aperiodic response has an infinite number of Poincare' points in its state space:

- i) an infinite number of points organized along an invariant curve (or topology): quasiperiodic response describing a beating like phenomena.
- ii) an infinite number of points that are not! organized along an invariant curve describe nonstationary response (and are a candidate for chaotic vibrations in deterministic systems).

Example:

Solve the system of two first order equations (equation 2), with $\gamma = 0.2, 0.28, 0.51, 0.46$,
 $\alpha = -1, \beta = 1, \delta = 0.3, \omega = 1.2$:

Poincare Map Procedure In Matlab

```
% File " poincare.m" , Poincare sampling procedure
clear all; close all; clc; format long;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% initial conditions %%%%%%%%%%%%%%%
t0=0;
tf=1000;
x0=[0.5 0]' ;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
w=1.2;                                % input frequency
Te=2*pi/w;                             % cycle time
RES=1000;                               % resolution - number of sample points per cycle
Tspan=[t0:(Te/RES):tf];
option=odeset ('RelTol', 1e-8, 'AbsTol',1e-8);      % option argument can be omitted
[t,X]=ode45 (@system1, Tspan, x0, option);
m=length(t);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% POINCARÉ SAMPLING %%%%%%%%%%%%%%%
per=0.5;                                % beginning of steady state
poincare_x=X(round(m*per):RES:m , 1) ;
poincare_y=X(round(m*per):RES:m , 2) ;
figure (1);
subplot (3,1,1) , plot (t,X(:,1));
xlabel ('t');ylabel ('x1');grid;

subplot (3,1,2) , plot (X(round(m*per):m,1),X(round(m*per):m,2));
xlabel ('x1');ylabel ('x2');grid;
subplot (3,1,3) , plot (poincare_x, poincare_y,'+');
xlabel ('x1');ylabel ('x2');grid;
```


(3) Forced vibration response: $\gamma = 0.2$

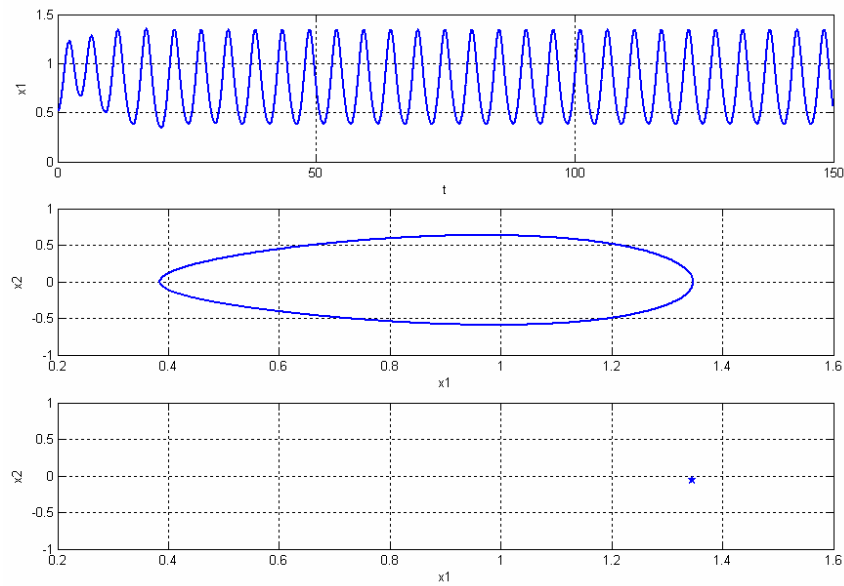


Figure 3: Forced vibration response: $\gamma = 0.2$, $\alpha = -1$, $\beta = 1$, $\delta = 0.3$, $\omega = 1.2$.

- a) Time series
- b) Phase plane at steady state ($120\text{sec} < t < 150\text{sec}$)
- c) Poincaré Map at steady state (period 1)

(4) Forced vibration response: $\gamma = 0.28$

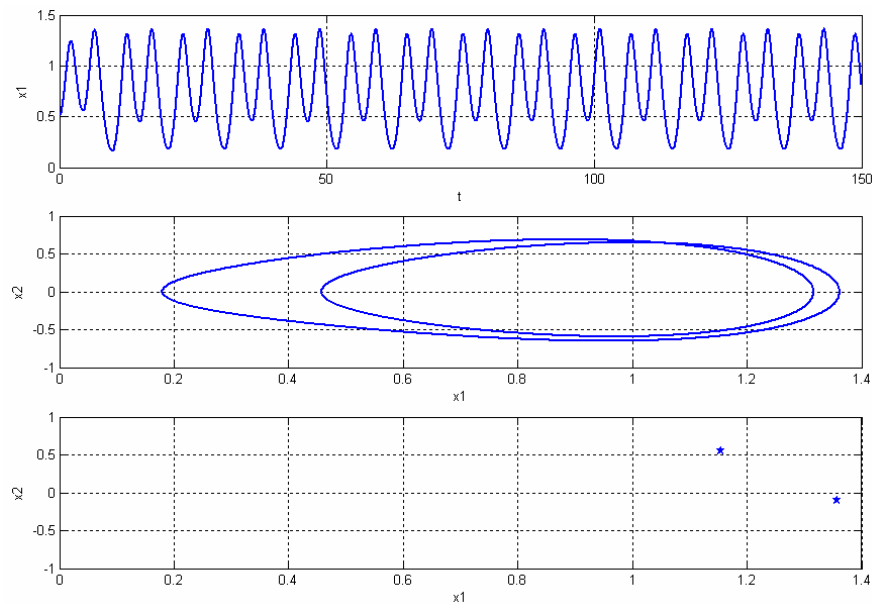


Figure 4: Forced vibration response: $\gamma = 0.28$, $\alpha = -1$, $\beta = 1$, $\delta = 0.3$, $\omega = 1.2$.

- a) Time series
- b) Phase plane at steady state ($120\text{sec} < t < 150\text{sec}$)
- c) Poincaré Map at steady state (period doubling)

(5) Forced vibration response: $\gamma = 0.51$

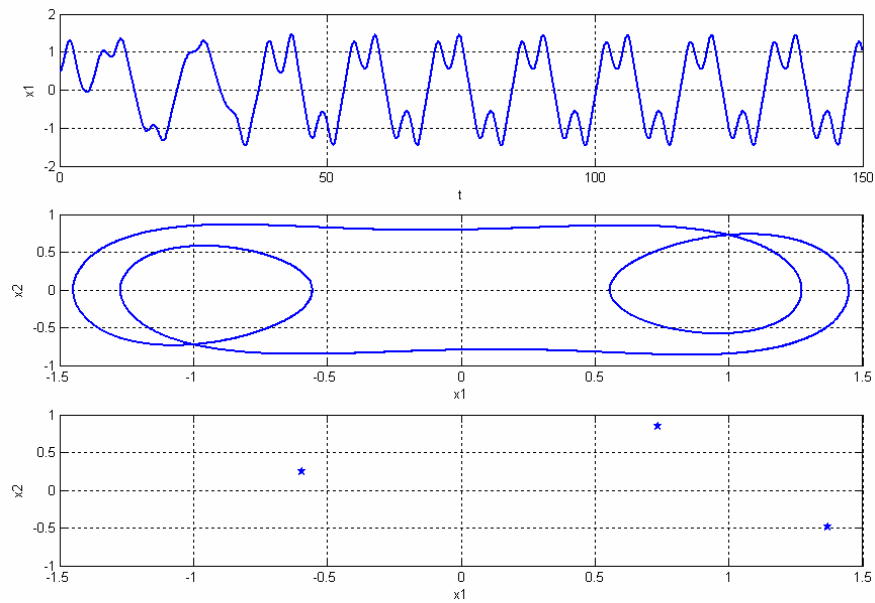


Figure 5: Forced vibration response: $\gamma = 0.51$, $\alpha = -1$, $\beta = 1$, $\delta = 0.3$, $\omega = 1.2$.

- a) Time series
- b) Phase plane at steady state ($120\text{sec} < t < 150\text{sec}$)
- c) Poincaré Map at steady state (period tripling)

(6). Forced vibration response: $\gamma = 0.46$

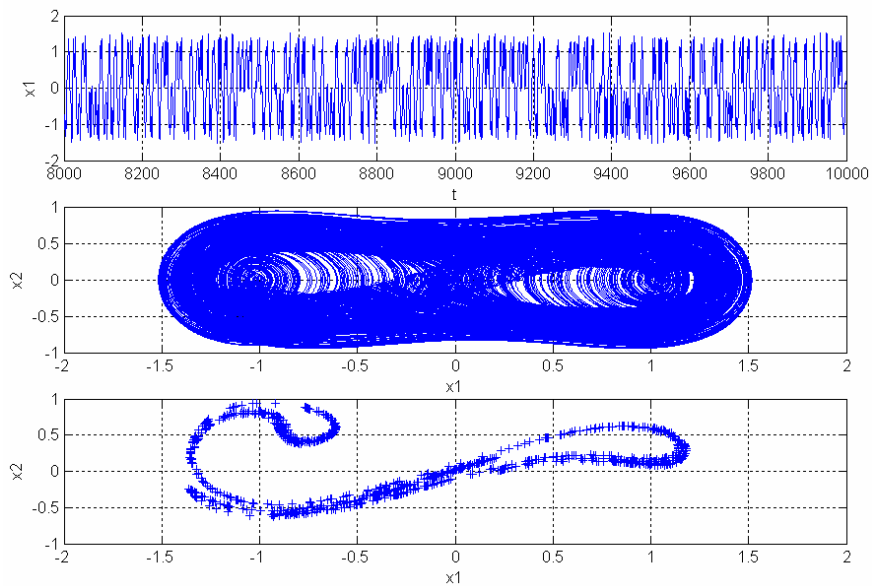


Figure 6: Forced vibration response: $\gamma = 0.46$, $\alpha = -1$, $\beta = 1$, $\delta = 0.3$, $\omega = 1.2$.

- a) Time series
- b) Phase plane at steady state ($5000\text{sec} < t < 10000\text{sec}$)
- c) Poincaré Map at steady state (aperiodic with 760 points)